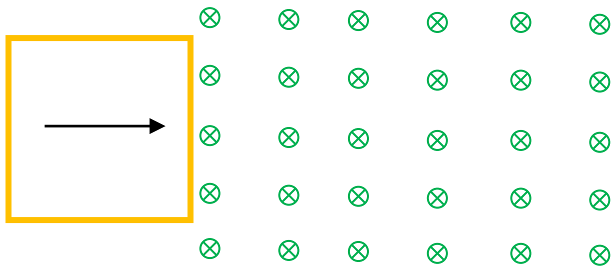
Teacher notes
Topic D

Conducting loop entering region of magnetic field – an exercise with exponentials.

A square conducting loop of side L is about to enter a large region of uniform magnetic field B directed into the page. The initial speed of the loop is u. The mass of the loop is m and its electric resistance is R.



(a) Explain why there will be an induced emf in the loop as it enters the region of magnetic field.
(b) Show that the induced emf is given by $\varepsilon = BvL$ where v is the speed the loop.
(c) Explain the direction of the magnetic force exerted on the loop.

The following data are available:

u = 14 m s^{-1}
ρ = 1.7×10^{-8} Ω m (resistivity of wire)
d = 8960 kg m^{-3} (density of wire)
B =0.12 T
m = 28 g
L = 0.25 m

(d)
(i) The speed of the loop as it enters the region of magnetic field is given by $v = ue^{-\alpha t}$ where $\alpha = \dfrac{B^2L^2}{mR}$. Show that α can be rewritten as $\alpha = \dfrac{B^2}{4d\rho}$ where ρ is the resistivity of the wire and d its density.
(ii) Evaluate α and state its unit.

(e)

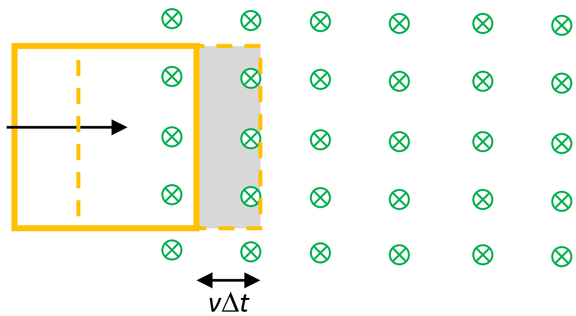
(i) The distance travelled is given by $x = \frac{u}{\alpha}(1 - e^{-\alpha t})$. Show that the speed of the loop when it is completely inside the magnetic field is about 8.0 m s^{-1}.

(ii) Determine the total thermal energy dissipated in the loop.

Answers

(a) The magnetic flux in the loop is increasing as more area is pierced by magnetic field lines. Faraday's law states that the induced emf is equal to the rate of change of magnetic flux.

(b) In time Δt, the loop moves forward a distance $v\Delta t$, and so the flux increases by $BLv\Delta t$. Hence the induced emf is $\varepsilon = \frac{\Delta\Phi}{\Delta t} = \frac{BLv\Delta t}{\Delta t} = BLv$.



(c) The flux is increasing and so the induced current will create a magnetic field opposite to the external magnetic field. This means the current is counterclockwise. Hence the magnetic force on the front part of the wire is to the left.

(d)

(i)

$$\frac{B^2L^2}{mR} = \frac{B^2L^2}{d(A4L)\rho\frac{L}{A}}$$

$$= \frac{B^2}{4d\rho}$$

Hence $v = ue^{-\frac{B^2L^2}{mR}t} = ue^{-\frac{B^2}{4\rho d}t}$.

(ii) $\alpha = \frac{B^2}{4d\rho} = \frac{0.12^2}{4\times 8960\times 1.7\times 10^{-8}} = 23.6\ \text{s}^{-1}$. αt must be dimensionless and so the unit is s^{-1}.

Explicitly,

$$[\alpha] = \frac{\text{T}^2}{\text{kg m}^{-3}\ \Omega\ \text{m}} = \frac{(\frac{\text{V}}{\text{m m s}^{-1}})^2}{\text{kg m}^{-3}\ \Omega\ \text{m}} = \frac{\text{V}^2}{\Omega}\frac{\text{m}^{-2}\ \text{s}^2}{\text{kg}} = \text{W}\frac{\text{m}^{-2}\ \text{s}^{-2}}{\text{kg}} = \frac{\text{N m}}{\text{s}}\frac{\text{m}^{-2}\ \text{s}^2}{\text{kg}} = \frac{\text{kg m s}^{-2}\ \text{m}}{\text{s}}\frac{\text{m}^{-2}\ \text{s}^2}{\text{kg}} = \text{s}^{-1}$$

(e)

(i) The loop will be entirely within the region of magnetic field when it moves a distance *L*.

$$x = L = \frac{u}{\alpha}(1 - e^{-\alpha t})$$

$$e^{-\alpha t} = 1 - \frac{L\alpha}{u} = 1 - \frac{0.25 \times 23.6}{14} = 0.578$$

Hence the speed is $v = ue^{-\alpha t} = 14 \times 0.578 = 8.09\text{ m s}^{-1}$.

(The time to enter is $-\alpha t = \ln 0.578 \Rightarrow t \approx 23\text{ ms}$.)

(ii) The energy lost is $\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2} \times 0.028 \times (14^2 - 8.09^2) = 1.8\text{ J}$. This is the energy that gets dissipated as thermal energy in the wire.

Calculus check: the thermal energy is also (*T* is the time to fully enter the magnetic field)

$$Q = \int_0^T P\,dt = \int_0^T \frac{\varepsilon^2}{R}dt = \int_0^T \frac{B^2L^2v^2}{R}dt = \int_0^T \frac{B^2L^2u^2e^{-2\alpha t}}{R}dt$$

$$= \frac{B^2L^2u^2}{R}\int_0^T e^{-2\alpha t}\,dt = -\frac{B^2L^2u^2}{R}\left.\frac{e^{-2\alpha t}}{2\alpha}\right|_0^T = \frac{B^2L^2u^2}{2\alpha R}(1 - e^{-2\alpha T}) \quad \text{but } \alpha = \frac{B^2L^2}{mR}$$

$$= \frac{m}{2}(u^2 - u^2e^{-2\alpha T}) = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

This energy is the result of someone doing (negative) work. Who is doing the work? There are no external forces pushing the loop. The magnetic force is at right angles to the velocity of the electrons, so it does zero work. So, who is doing the work? (Hint: to move a charge Δq across a potential difference *V* requires work $V\Delta q$.)

Math details

$$F = BIL = B\frac{\varepsilon}{R}L = B\frac{BvL}{R}L = \frac{B^2L^2}{R}v$$

$$m\frac{dv}{dt} = -\frac{B^2L^2}{R}v \Rightarrow v = ue^{-\frac{B^2L^2}{mR}t} = ue^{-\alpha t}$$

$$\frac{dx}{dt} = ue^{-\frac{B^2L^2}{mR}t} \Rightarrow x = \frac{u}{\frac{B^2L^2}{mR}}(1 - e^{-\frac{B^2L^2}{mR}t}) = \frac{u}{\alpha}(1 - e^{-\alpha t})$$